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STRESSED-STRAINED STATE OF A HOLLOW CYLINDER IN THERMAL DIFFUSION OF CARBON INTO ITS WALL

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Using the finite-element method, we reduce the thermoelasticity problem to a system of difference equations solvable by a factorization method. The influence of the swelling of the crystal lattice of steel during the penetration of carbon into it on the stressed-strained state is considered.

Saturation of structural elements with carbon diffusing from a body surface can be both forced with the aim of improving the mechanical properties of a material (cementation) and independent , for example, transfer of carbon by a sodium coolant in power installations. On exposure of a steel structure to carbon, the chemical composition and physicomechanical and thermophysical characteristics of the steel structure in the saturation depth vary, which leads to the need for evaluating the stressed-strained state of the part in the context of the mechanics of inhomogeneous structures.

Moreover, the penetration of carbon into the crystal lattice of a metal causes its swelling [1], which creates the field of supplementary (technological) stresses that depend on the concentration level of a diffusing substance. In the case of a short-term regime of carbonization (the process of cementation), relaxation phenomena can be neglected. In long-term processes of saturation with carbon (in the case of small powers of carbonization sources), the growth rate of the stresses and their relaxation become comparable. In the case of the mean power of the sources of saturation with carbon, the relaxation processes in the first approximation can be neglected.

We consider a hollow cylinder with inner radius $r = \tilde{R}_1$ and outer radius $r = \tilde{R}_2$, which is located in a polar-symmetric thermal field under the action of carbon on its outer surface. The distribution of the concentration $c = c(r, t)$ $(0 \le c \le 1,\ 1$ wt.%) of the diffusing carbon at any instant of time t is determined by the solution of the coupled thermal diffusion problem [2]

$$\operatorname{div}(D \operatorname{grad} c) = \frac{\partial c}{\partial t}, \tag{1}$$

$$\operatorname{div}(\lambda \operatorname{grad} T) = 0 \tag{2}$$

with the following initial and boundary conditions:

$$t = 0,\ c = 0 \ \text{ for } \ \tilde{R}_1 \le r < \tilde{R}_2, \tag{3}$$

$$c = 0 \ \text{ for } \ r = \tilde{R}_1,\ c = c(t) \ \text{ for } \ r = \tilde{R}_2 \ \ (t \ge 0), \tag{4}$$

$$T = \tilde{T}_1 \ \text{ for } \ r = \tilde{R}_1,\ T = \tilde{T}_2 \ \text{ for } \ r = \tilde{R}_2. \tag{5}$$

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Here $D = D_0 \exp(-Q/(RT))$ and $\lambda = \lambda(c, T)$. The source of carbon saturation is of the mean power; therefore its concentration reaches the limiting value c = 1 wt.% at $r = \tilde{R}_2$ in conformity with the law [3] $c(t) = B_0 \frac{k(T)+1}{k(T)} \exp(-\frac{Q}{RT}) t^n$ and subsequently remains constant, i.e., for now we do not consider the inverse kinetics. A heat-conduction equation is selected in the form of Eq. (2), since the diffusion rate is much lower than the velocity of heat propagation and therefore the thermal regime is assumed to be stationary.

For simultaneous solution of Eqs. (1) and (2) we use the method of temporal layers, assuming that within the limits of a time step the temperature field remains constant but varies from step to step. Replacing the boundary-value problems (1), (3), (4), and (2), (5) by variational ones that are equivalent to them and applying the finite-element method, we obtain the system of difference equations for determining the nodal values of the temperature and concentration of the carbon T_i and c_i:

$$A_1 \frac{dc_i}{dt} + A_2 \frac{dc_{i-1}}{dt} + A_3 \frac{dc_{i+1}}{dt} + A_4 c_i - A_5 c_{i+1} - A_6 c_{i-1} = 0, \quad i = 2, ..., N; \tag{6}$$

$$a_i T_i + b_i T_{i+1} + d_i T_{i+2} = 0, \quad i = 1, ..., N-1, \tag{7}$$

where

$$A_1 = \frac{1}{12(r_{i+1} - r_i)} [r_{i+1}^3 + r_{i+1}^2 r_i - 5 r_{i+1} r_i^2 + 3 r_i^3] + \frac{1}{12(r_i - r_{i-1})} \times$$

$$\times [r_{i-1}^3 + r_{i-1}^2 r_i - 5 r_{i-1} r_i^2 + 3 r_i^3];$$

$$A_2 = \frac{1}{12} [r_i^2 - r_{i-1}^2]; \quad A_3 = \frac{1}{12} [r_{i+1}^2 - r_i^2];$$

$$A_4 = \frac{D_i^*}{2} \frac{r_{i+1} r_i}{r_{i+1} r_i} + \frac{D_{i-1}^*}{2} \frac{r_i + r_{i-1}}{r_i - r_{i-1}}; \quad A_5 = \frac{D_i^*}{2} \frac{r_{i+1} + r_i}{r_{i+1} - r_i}; \quad A_6 = \frac{D_{i-1}^*}{2} \frac{r_i + r_{i-1}}{r_i - r_{i-1}},$$

$$i = 2, ..., N;$$

$$a_i = -\frac{\lambda_i^*}{2} \frac{r_{i+1} + r_i}{r_{i+1} - r_i}; \quad b_i = \frac{\lambda_i^*}{2} \frac{r_{i+1} + r_i}{r_{i+1} - r_i} + \frac{\lambda_{i+1}^*}{2} \frac{r_{i+2} + r_{i+1}}{r_{i+2} - r_{i+1}}; \quad d_i = -\frac{\lambda_{i+1}^*}{2} \frac{r_{i+2} + r_{i+1}}{r_{i+2} - r_{i+1}},$$

$$i = 1, ..., N-1;$$

N is the number of the elements of splitting of the tube cross section; λ_i^* and λ_{i+1}^* and D_i^* and D_{i+1}^* are the mean values of the coefficients of thermal conductivity and diffusion in the elements with numbers i and $i + 1$, respectively.

To solve the problem of determining the penetration time of the diffusing substance into the tube wall, it is necessary to solve Eqs. (6) and (7) with the following boundary conditions:

$$c_1 = 0, \quad c_s = c(t), \tag{8}$$

$$T_1 = \tilde{T}_1, \quad T_s = \tilde{T}_2, \tag{9}$$

where $s = N + 1$ is the number of nodes.

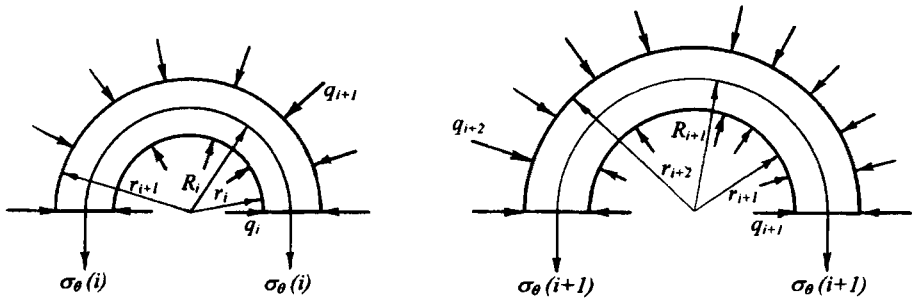

Fig. 1. Two contacting layers of the transverse cross section of a cylinder.

Using the Crank–Nicholson difference scheme [4], we can solve the system of ordinary differential equations (6). We integrate each of Eqs. (6) with respect to time over the interval $t_j \le t \le t_{j+1}$, assuming the invariance of the diffusion coefficient within the limits of the time step $\Delta t = t_{j+1} - t_j$ ($D_i^* = \text{const}$, $D_{i-1}^* = \text{const}_1$). Taking into account that $\frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} c_k dt \approx \frac{1}{2}(c_k^j + c_k^j + 1) \approx \frac{1}{2}(c_k^j + c_k^{j+1})$, $k = i-1, i, i+1$, we obtain

$$\left(\frac{2A_1}{\Delta t} + A_4\right)c_i^{j+1} + \left(\frac{2A_2}{\Delta t} - A_6\right)c_{i-1}^{j+1} + \left(\frac{2A_3}{\Delta t} - A_5\right)c_{i+1}^{j+1} =$$

$$= \left(\frac{2A_1}{\Delta t} - A_4\right)c_i^j + \left(\frac{2A_2}{\Delta t} + A_6\right)c_{i-1}^j + \left(\frac{2A_3}{\Delta t} + A_5\right)c_{i+1}^j,$$

$i = 2, ..., N$; $j = 0, ..., M$; M is the number of time steps; $c_1^0 = c_2^0 = ... = c_{N+1}^0 = 0$; $c_1^j = 0$, $c_{N+1}^j = c(t_j)$ at $j = 1, 2, ..., M$.

The difference equations are solved by the factorization method [4].

Having determined the distributions $c = c(r, t)$ and $T = T(r)$ [2], we solve the thermoelasticity problem.

We take the laws of distribution of the Young modulus E, the linear expansion coefficient α, and the Poisson coefficient μ to be dependent on the temperature T and the concentration of carbon c in the form $E = E_0(T)(1 - \beta(T)c)$, $\alpha = \alpha_0(T)(1 - \gamma(T)c)$, and $\mu = \mu_0(T)(1 - \delta(T)c)$.

In order to solve the thermoelasticity problem, we split the cylinder cross section into N layers and consider two adjacent layers with the numbers i and $i + 1$ (Fig. 1). We will use the following notation: R_i and R_{i+1} for the mean radii of the layers; $\sigma_\theta(i)$ and $\sigma_\theta(i + 1)$ for the circumferential stresses in the layers; q_{i+1} and T_{i+1} for the contact pressure of the layers and for the temperature in the zone of contact. The distribution of the temperature and concentration of the carbon inside each layer is approximated by the linear functions

$$t_i(r) = T_i + \frac{T_{i+1} - T_i}{h_i}(r - r_i), \quad c_i(r) = c_i + \frac{c_{i+1} - c_i}{h_i}(r - r_i), \quad r_i \le r \le r_{i+1};$$

$$h_i = r_{i+1} - r_i, \quad i = 1, ..., N; \quad N + 1 = s.$$

Here c_i and c_{i+1} are the nodal values of the carbon concentration.

In what follows, we take the condition that the cylinder can freely diffuse in the longitudinal direction and at the same time can have a bottom or not.

The circumferential stresses in the layers can be determined by summing their values that are attributable to the pressure and are found by virtue of the thin-walled layers according to the membrane theory of shells with the stresses from a temperature drop over the cylinder-wall thickness and stresses caused by the swelling of the crystal lattice [1]:

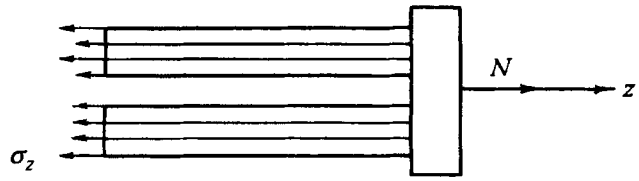

Fig. 2. Cut-off portion of a cylinder.

$$\sigma_\theta(i) = \frac{q_i(2R_i - h_i) - q_{i+1}(2R_i + h_i)}{2h_i} + \frac{E_i\alpha_i(T_i - T_{i+1})}{1-\mu_i}\frac{r-R_i}{h_i} +$$

$$+ \frac{E_i\sum_{k=1}^{m}\lambda_k(c_i^k - c_{i+1}^k)}{1-\mu_i}\frac{r-R_i}{h_i}, \tag{10}$$

$r_i \le r \le r_{i+1}$, $i = 1, \ldots, N-1$, $q_1 = 0$, $q_{N+1} = 0$, E_i, and μ_i are the element-mean values of the Young modulus and of the Poisson coefficient; λ_k are the swelling parameters.

To determine the axial stresses σ_z, we take, as a basis, the hypothesis for the plane deformation $\varepsilon_z = \xi = \text{const}$.

Let us consider the equilibrium of the cut-off portion of the cylinder (Fig. 2). Projecting all the forces that act on the cylinder onto the z axis, we obtain

$$\int_{r_1}^{r_2}\sigma_z\,rdr = N = 0 \quad \text{or} \quad \sum_{i=1}^{N}\int_{r_i}^{r_{i+1}}\sigma_z(i)\,rdr = 0. \tag{11}$$

According to Hooke's law, the axial deformation of the cylinder is as follows:

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \mu\sigma_\theta] + \alpha t(r) + \sum_{k=1}^{m}\lambda_k c^k(r) = \xi.$$

Whence

$$\sigma_z = E\xi + \mu\sigma_\theta - \alpha E t(r) - E\sum_{k=1}^{m}\lambda_k c^k \tag{12}$$

or for each layer

$$\sigma_z(i) = E_i\xi + \mu_i\sigma_\theta(i) - \alpha_i E_i t_i(r) - E_i\sum_{k=1}^{m}\lambda_k c_i^k(r), \quad i = 1, \ldots, N.$$

Here E_i, α_i, and μ_i are the mean values of the Young modulus, the linear expansion coefficient, and the Poisson coefficient over the layers. Then condition (11) gives

$$A\xi + B = 0, \tag{13}$$

where

TABLE 1. Values of the Mechanical and Thermophysical Characteristics of the Material

T, K	Characteristics	$\tilde{n}$					
		0	0.2	0.4	0.6	0.8	1
873	λ	0.330	0.237	0.207	0.187	0.177	0.174
	E	127	126	125	124	122	121
	α	1.59	1.59	1.59	1.58	1.58	1.58
973	λ	0.343	0.247	0.217	0.197	0.187	0.186
	E	117	116	115	114	113	111
	α	1.65	1.64	1.64	1.64	1.63	1.63

$$B=\sum_{i=1}^{N}\left(\frac{r_{i+1}^2-r_i^2}{4h_i}\mu_i\left(q_i\left(2R_i-h_i\right)-q_{i+1}\left(2R_i+h_i\right)\right)+\frac{\mu_i}{1-\mu_i}E_i\alpha_i\frac{T_i-T_{i+1}}{h_i}\times\right.$$

$$\left.\times\left[\frac{r_{i+1}^3-r_i^3}{3}-R_i\frac{r_{i+1}^2-r_i^2}{2}\right]\right)-$$

$$-\sum_{i=1}^{N}\alpha_i E_i\left[T_i\frac{r_{i+1}^2-r_i^2}{2}+\frac{T_{i+1}-T_i}{h_i}\left(\frac{r_{i+1}^3-r_i^3}{3}-r_i\frac{r_{i+1}^2-r_i^2}{2}\right)\right]+$$

$$+\sum_{i=1}^{N}\frac{\mu_i}{1-\mu_i}E_i\sum_{k=1}^{m}\lambda_k\left(c_i^k-c_{i+1}^k\right)\frac{1}{h_i}\left[\frac{r_{i+1}^3-r_i^3}{3}-R_i\frac{r_{i+1}^2-r_i^2}{2}\right]-$$

$$-\sum_{i=1}^{N}E_i\sum_{k=1}^{m}\lambda_k\sum_{n=0}^{k}\overset{\cup}{C}_k^n c_i^n\left(c_{i+1}-c_i\right)^{k-n}\left[\frac{h_i^2}{k-n+2}+r_i\frac{h_i}{k-n+1}\right],$$

$$A=\sum_{i=1}^{N}E_i\frac{r_{i+1}^2-r_i^2}{2},\quad i=1,\ldots,N;$$

$\overset{\cup}{C}_k^n$ are the binomial coefficients.

To find the contact pressures q_i (i = 2, ..., N), we use the condition for the consistency of deformations. This will require the equality of the radial displacements $u(r)$ of the cylinder layers in the zone of their conjugation: $u_i = u_{i+1}$ at $r = r_{i+1}$ (i = 1, ..., $N-1$), where

$$u_i(r)=\frac{r}{E_i}\left[\sigma_\theta(i)-\mu_i\sigma_z(i)\right]+\alpha_i rt_i(r)+r\sum_{k=1}^{m}\lambda_k c^k(r).$$

As a result, we obtain the system of difference equations

$$a_i q_i+b_i q_{i+1}+c_i q_{i+2}=\tilde{f}_i+f_i^*\xi, \tag{14}$$

where

$$a_i=\frac{\left(1-\mu_i^2\right)\left(2R_i-h_i\right)}{2E_i h_i};$$

$$b_i = -\frac{(1-\mu_i^2)(2R_i + h_i)}{2E_i h_i} - \frac{(1-\mu_{i+1}^2)(2R_{i+1} - h_{i+1})}{2E_{i+1} h_{i+1}};$$

$$c_i = \frac{(1-\mu_{i+1}^2)(2R_{i+1} + h_{i+1})}{2E_{i+1} h_{i+1}};$$

$$\tilde{f}_i = \frac{(1+\mu_{i+1})\alpha_{i+1}(T_{i+1} - T_{i+2})(r_{i+1} - R_{i+1})}{h_{i+1}} -$$

$$-\frac{(1+\mu_i)\alpha_i(T_i - T_{i+1})(r_{i+1} - R_i)}{h_i} +$$

$$+\alpha_{i+1}(1+\mu_{i+1})T_{i+1} - \alpha_i(1+\mu_i)T_{i+1} +$$

$$+\frac{1+\mu_{i+1}}{h_{i+1}}\sum_{k=1}^{m}\lambda_k(c_{i+1}^k - c_{i+2}^k)(r_{i+1} - R_{i+1}) -$$

$$-\frac{1+\mu_i}{h_i}\sum_{k=1}^{m}\lambda_k(c_i^k - c_{i+1}^k)(r_{i+1} - R_i) + \sum_{k=1}^{m}\lambda_k c_{i+1}^k(\mu_{i+1} - \mu_i);$$

$$f_i^* = -(\mu_{i+1} - \mu_i), \quad i = 1, ..., N-1.$$

Solving this system by the factorization method [4] with the conditions

$$q_1 = 0, \quad q_s = 0, \tag{15}$$

that mean the absence of the pressure on the inner and outer cylinder surface, we obtain q_i (i = 2, ..., N) in the form

$$q_i = \tilde{q}_i + q_i^* \xi, \tag{16}$$

where $\tilde{q}_i$ is the solution of problem (14) and (15) that corresponds to the right-hand side $\tilde{f}_i$ of Eq. (14); q_i^* is the solution of the same problem but with the right-hand side f_i^* in Eq. (14); ξ is unknown. Substituting Eq. (16) into (13), we find first ξ and then q_i. Here it should be noted that if no allowance is made for the dependence of the Poisson coefficient on the carbon concentration and it (the Poisson coefficient) is considered to be constant over the cylinder cross section, then $f_i^* = 0$ and $q_i^* = 0$, and it is not necessary to solve Eq. (13) for finding the contact pressures q_i.

Thus, having obtained the pressures q_i and by means of them the radial stresses $\sigma_r(i) = -q_i$, from formula (10) we find the distribution of the stress σ_θ and then, after the determination of $\varepsilon_z = \xi$ from Eq. (13), we find the distribution of σ_z from formula (12).

Consider an example of calculation that is carried out for the cylinder with geometric dimensions $\tilde{R}_1$ = 94.5 mm and $\tilde{R}_2$ = 99.5 mm; the temperature on the inner cylinder surface is T_1 = 893 K and on the outer surface, $\tilde{T}_2$ = 973 K; the material is 1X18H10T-grade steel. In the calculations, we took the following data, obtained on the basis of [3, 5]: the diffusion coefficient $D = D_0 \exp(-k_0/T)$, where k_0 = 11,150 K and D_0 = 7.4443 mm^2/h. Table 1 presents the dependence of the thermal conductivity coefficient $\lambda(c, T)$ (J/mm·h·K) on the concentration of carbon c and the temperature T, and also the values of the Young modulus E (GPa) and the linear expansion coefficient $\alpha \cdot 10^5$ (1/deg), which is taken hypothetically. The Poisson coefficient is μ =

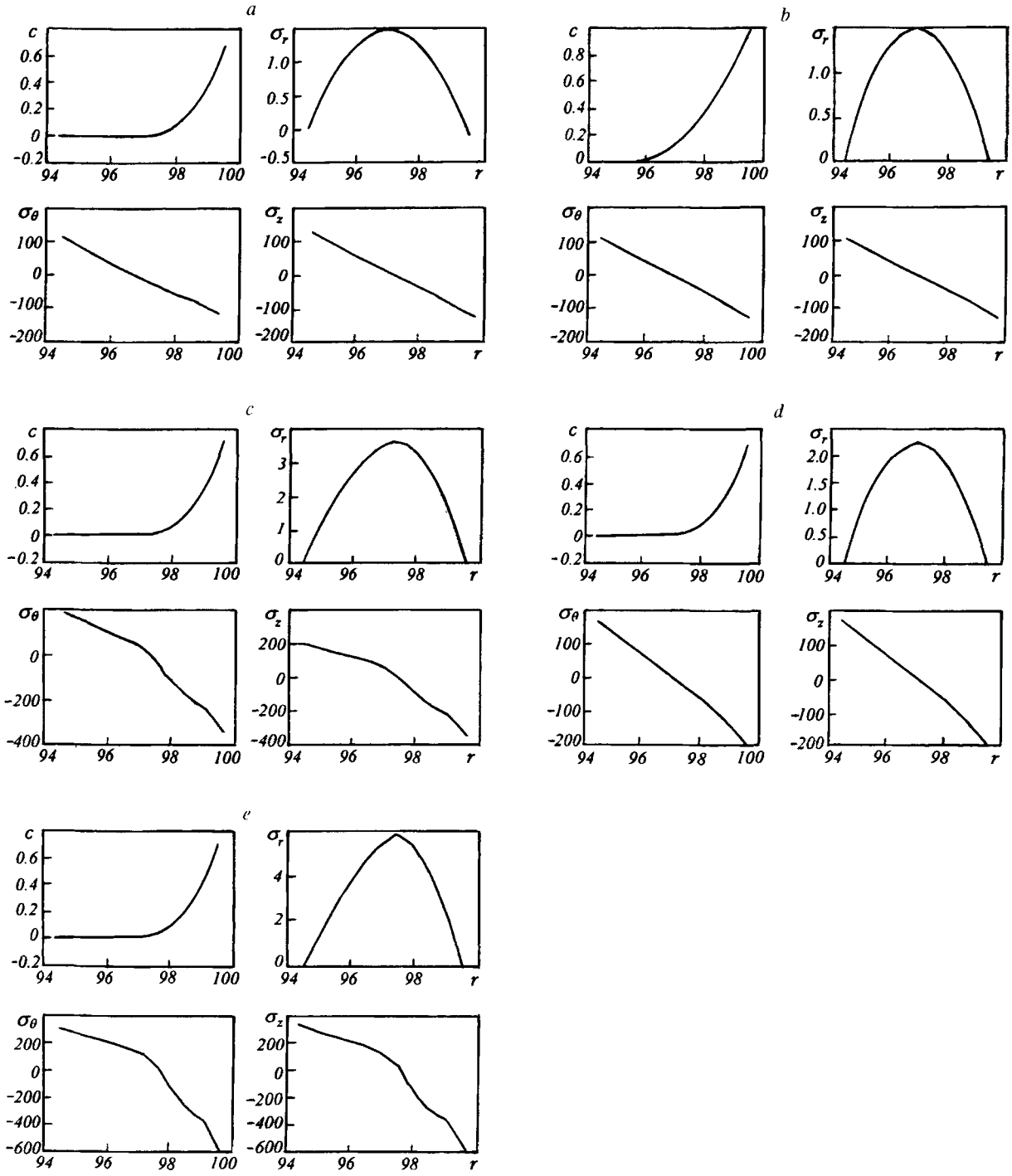

Fig. 3. Graphs of the distribution of the stresses (MPa) and the concentration (percentage) of carbon over the cylinder radius: a and d) without allowance for the swelling effect after 10,000 h; b) same after 25,000 h; c, e) with allowance for the swelling effect after 10,000 h. r, mm.

0.3; B_0 = 2340.87, $k(T)$ = 0.775 (for T = 973 K); n = 0.52. The swelling parameters λ_i (i = 1, ..., 5) are taken just the same as for 45Kh-grade steel and have the following values: λ_1 = 0.0158; λ_2 = –0.0855; λ_3 = 0.2143; λ_4 = –0.2422; λ_5 = 0.1024. The number of splitting elements was taken to be equal to N = 10. The time step

was 500 h. The given parameters of splitting turned out to be sufficient for reaching the required accuracy. A decrease in the splitting steps led to a change of less than 0.5% in the results of calculations.

Figure 3a and b presents the results of the calculations without considering the effect of swelling of the crystal lattice. The results of the calculations obtained with allowance for the swelling of the crystal lattice are given in Fig. 3c. In these cases, the material was taken to be thermally sensitive and with properties dependent on the carbon concentration. Figure 3d and e illustrates the graphs of the distribution of the stresses obtained on the assumption that the Young modulus, the linear expansion coefficient, and the Poisson coefficient are independent of the temperature and the concentration of the carbon. In the calculations they were taken to be as follows: E = 215 GPa, $\alpha = 1.15 \cdot 10^{-5}$ 1/deg, and μ = 0.3.

Thus, the method presented enables one to obtain the solution of the coupled problem of thermal diffusion and the thermoelasticity problem for hollow cylinders with their surface carbonization. The investigations carried out have shown that the thermal sensitivity of the material and the effect of swelling of the crystal lattice exert an influence on the level of stresses. Here it turned out to be possible to displace the most stressed point in the cylinder cross section from the inner surface to the outer one subjected to the carbonization. On the basis of the aforesaid, a conclusion can be drawn on the need to allow for the indicated factors in carrying out investigations.

NOTATION

$\tilde{R}_1$, inner radius of the hollow cylinder; $\tilde{R}_2$, outer radius of the hollow cylinder; r, polar coordinate; t, time; T, temperature; c, relative concentration of carbon; D, diffusion coefficient; Q, activation energy; R, gas constant; D_0, constant for the given material that characterizes the diffusion coefficient; λ, thermal conductivity coefficient; T_1 and T_2, temperatures on the inner and outer cylinder surfaces, respectively; r_i, polar coordinates of the grid nodes; c_i, nodal values of the carbon concentration; λ_i^* and D_i^*, element-mean splitting values of the coefficients of thermal conductivity and diffusion; σ_r , σ_θ, and σ_z, radial, circumferential, and axial stresses, respectively; B_0, constant that characterizes the change in the carbon concentration.

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